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LETTER TO THE EDITOR

Discussion of "Stochastic Approaches for Damage Evolution in Standard and Non-standard Continua", *Int. J. Solids Structures*, Vol. 32, No. 8/9, pp. 1149–1160 (1995).

In their paper, Carmeliet and de Borst attempt to include the effects of heterogeneity on material response. The paper recognizes that "damage evolution in quasi-brittle materials is a complex process in which heterogeneity plays an important role." For this reason, stochastic distributions of certain material properties are considered. In particular, the initial damage threshold level K_0 is assumed to be a random field with a Gaussian autocorrelation function. The relevant autocorrelation length θ is the first length introduced in their work. As correctly (and obviously) mentioned by the authors, the stochastic approach does not resolve the issue of change of character in the governing differential equation in the softening regime. For this reason an additional length scale l is introduced by assigning non-local properties to the relevant damage variable D. The present Letter to the Editor addresses (a) important inconsistencies present in the formulation, (b) the physical and mathematical interpretation of the two length scales introduced, and (c) the importance of surface effects for the type of problems considered. In this perspective, it is shown herein that the paper introduces a redundant formulation, physically unreasonable, and pinpoints towards the wrong direction of research in the subject area.

The length scale present in non-local, gradient and viscous continuum theories represents, in general, the spatial "range" of significant mechanical interactions among nearby points. A conjugate length may also be considered, for example by assigning non-local properties to $1-D$, instead of D. In any case, material microstructure is decisive on the magnitude of that length (if this was not the case, then a universal length scale would exist). Thus in such theories, the relevant length scale l is directly related to the material microstructure. Heterogeneity is in all pragmatic terms the realization of (micro)structure. Thus, the two length scales of Carmeliet and de Borst (1995) should be related to each other, and we show why and how in the sequence.

For demonstration purposes, we consider a (strictly) uniaxial tension problem. As is known, i.e. Sluys and de Borst (1994), the non-local strain measure, $\bar{\varepsilon}$ or the rate $\dot{\bar{\varepsilon}}$, for a rate formulation, can be expressed, through series expansion as (the notation is similar to the one in the paper discussed)

$$\dot{\bar{\varepsilon}} = \dot{\varepsilon}(x) + bl^2 \frac{\partial^2 \dot{\varepsilon}(x)}{\partial x^2} \tag{1}$$

where b depends on the weight function chosen for the non-local "convolution" operation. As an example, for step-type (uniform) weight function $b = 1/24$. For the linear softening case, Fig. 1 in Carmeliet and de Borst (1995), the rate of damage $\dot{D}$ is related to $\dot{\bar{\varepsilon}}$ as

$$\dot{D} = \frac{\varepsilon_u K_0}{\varepsilon_u(\varepsilon_u - K_0)\bar{\varepsilon}^2}\dot{\bar{\varepsilon}}. \tag{2}$$

By substituting eqns (1) and (2) in eqn (2) of Carmeliet and de Borst (1995), we obtain

$$\dot{\sigma} = \left(1 - D_0 - \frac{\varepsilon_u K_0 \varepsilon_0}{(\varepsilon_u - K_0)\bar{\varepsilon}^2}\right) C\dot{\varepsilon} - \frac{\varepsilon_u K_0 \varepsilon_0}{(\varepsilon_u - K_0)\bar{\varepsilon}^2} C b l^2 \frac{\partial^2 \dot{\varepsilon}}{\partial x^2}. \quad (3)$$

The damage threshold K_0 is considered to be a random field. Thus in the above rate equation, the expression, termed herein as S,

$$S = \frac{\varepsilon_u K_0 \varepsilon_0}{(\varepsilon_u - K_0)\bar{\varepsilon}^2} C \quad (4)$$

is the stochastic "input." It can be readily seen that an equivalent formulation calls for the modulus C being the stochastic input. The term $1 - D_0$ in eqn (3) does not alter the second order statistics, the "juice" of the problem, but simply the mean in the non-gradient term; the shift in this mean may or may not have a physical interpretation but this is irrelevant to the present discussion. Such an equivalent formulation seems to be more tractable numerically and experimentally. Perhaps this point was not noticed by the authors. The following important points should be mentioned:

- It is interesting that $\bar{\varepsilon}^2$ appears in the denominator in eqn (4). Since $\bar{\varepsilon}$ increases with deformation, this implies that the variance of S decreases continuously, and rapidly (proportionally to $1/\bar{\varepsilon}^4$). This implies, for the *equivalent formulation* mentioned above, that the *variance* of C decreases with deformation. This can be interpreted physically as damage reduction or crack closing under uniaxial tension! Thus, the competition between this (nonphysical) reduction in variance and increase in D as a function of $\bar{\varepsilon}$ seems to be more important than the interaction of the two length scales mentioned by the authors. Note that although it is difficult to determine the properties of $\bar{\varepsilon}$ analytically, since it is integrated in time it is smoothed out, especially with respect to $\dot{\varepsilon}$, and K_0, the statistical properties of the latter being fixed.
- The definition of the correlation distance as "the length over which the autocorrelation coefficient function drops to a small value, say e^{-1}", is quite unique. This happens to be only true for the exponential autocorrelation function, since $\int_0^\infty \mathrm{Exp}[-x/c]\,dx = c$. It is not true for the Gaussian function used by the authors; a straightforward derivation or any text book on correlation statistics can reveal this, i.e. Yaglom (1987). As a result the values used for correlation distances are wrong by a factor $\sqrt{\pi}/2$, in all the computational results. This implies an error of about 12.5% in the length scale used. The numerical values given indicate clearly that the incorrect expression given in the paper ($d = \theta/\sqrt{2}$) is not due to a misprint, for example the values given, $d = 5$ mm, $\theta = 7$ mm, correspond to this incorrect relation. The correct expression is given below. It is remarkable that the work of Carmeliet and de Borst (1995) was performed without even proper understanding of the physics and mathematics of basic (for the subject area) concepts as the correlation distance.
- Another very inconsistent point is the process followed to avoid "defining statistical boundary conditions or considering boundary layer effects." The problem is not to generate a random field that has the same properties near and far from boundaries, as implied by the authors. Although most random field generation codes do generate fields without boundary effects, what the authors claim is misleading, to say the least. Boundary problems are present in their results anyway. As a simple illustrative example, consider a deterministic boundary condition where the displacement is specified. Then on the boundary the variance of the displacement is obviously zero. Far from the boundary the displacement will have a non-zero variance. The transition zone from zero variance of displacement at the boundary to a finite value, defines the boundary effects, and their extent (depth) on displacement in this case. Usually such effects extend to about 3–5 correlation distances, and the effects on

strains, important for fracture, can be quite adverse. We refer to Frantziskonis (1995) for extensive details on such problems. So, although boundary effects are present, nothing is mentioned in the presentation of the numerical results.
- If K_0 is stationary in the strict sense, or K_0 is stationary in the wide sense and normal, then S is stationary, Papoulis (1991). In these cases the statistics of S can be calculated, Yaglom (1987). In Carmeliet and de Borst (1995) a non-Gaussian field for K_0 is considered, and it is not clear if it is considered as stationary in the strict or wide sense. If K_0 is stationary in the wide sense, then S may be non-stationary. This would imply, for the equivalent formulation mentioned above, that C is non-stationary, and this seems hard to justify on physical grounds.

From the above, we can only conclude that the formulation of Carmeliet and de Borst (1995) is based on "analogical" mechanics and "analogical" statistics.

So now let us address the problem of the two length scales. As mentioned above the formulation is equivalent to assigning random field properties to C, with a shift in the mean value for the non-gradient part. Under uniaxial conditions, for an infinite medium (thus excluding boundary effects) this will render the strain to be a stochastic field (under specified stress it is straightforward to show that the strain will be stationary, i.e. from eqn 1 in Carmeliet and de Borst (1995). Under specified displacement, for an infinite medium the derivation is more involved, but although it can be shown that the strain remains stationary, however, this is not crucial in this discussion. In any case we start by considering the strain field to be stationary or with stationary increments, the latter being a much more general case. The situation will change as localization may onset, and this is discussed subsequently.

Since the strain is non-uniform as a result of micro-structure, it can be considered as a micro-deformation gradient, as explained in a lucid fashion by Mindlin (1964). Then the macro-strain, E is some function of ε, i.e. in general, $E = E(\varepsilon)$. Let m be the expected value of ε, thus $\langle\varepsilon\rangle = m$. Assuming (see following discussion) that the fluctuations of ε are small, if $E'(m)$ is $\partial E(\varepsilon)/\partial\varepsilon$ evaluated at m we have the Taylor expansion

$$E(\varepsilon) = E(m)+E'(m)(\varepsilon-m)+\tfrac{1}{2}E''(m)(\varepsilon-m)^2+\mathrm{O}[.]^3. \tag{5}$$

From this expansion and since $\varepsilon = \varepsilon(x)$ is defined in the x-domain, taking the ensemble average in eqn (5), up to second order terms, since the mean of $\varepsilon-m$ is zero and the mean of $(\varepsilon-m)^2$ is the variance of ε, denoted as $\mathrm{Var}(\varepsilon)$, and $E''(m)$ is now dependent on x, by considering ε to be ergodic, we have

$$\langle E\rangle = E(m)+\tfrac{1}{2}\mathrm{Var}\,(\varepsilon)\left\langle\left\{\left[\frac{\partial\varepsilon}{\partial x}\right]^{-3}\left[\frac{\partial\varepsilon}{\partial x}\frac{\partial^2 E}{\partial x^2}-\frac{\partial^2\varepsilon}{\partial x^2}\frac{\partial E}{\partial x}\right]\right\}_{\varepsilon=m}\right\rangle. \tag{6}$$

Differentiable random functions are characterized by twice differentiable autocorrelation functions, Yaglom (1987). For Gaussian autocorrelation of ε considered in the following this is true, as is the case for most autocorrelation functions of stationary processes, and for several processes with stationary increments. For function ε being sufficiently smooth with respect to x at $\varepsilon = m$ the last term in the above equation can be considered small as compared to the term preceding it. Although in a general statistical analysis this assumption is not necessary, for the analytical illustration purposes intended herein this is, in general, sufficient. Then,

$$\langle E\rangle \simeq E(m)+\tfrac{1}{2}\mathrm{Var}(\varepsilon)\left\langle\left\{\left[\frac{\partial\varepsilon}{\partial x}\right]^{-2}\frac{\partial^2 E}{\partial x^2}\right\}_{\varepsilon=m}\right\rangle. \tag{7}$$

For ε with twice differential autocorrelation, we have, Vanmarcke (1983), Yaglom (1987),

$$\left\langle \left(\frac{\partial \varepsilon}{\partial x}\right)^2 \right\rangle = -\left.\frac{\partial^2 P_\varepsilon(\tau)}{\partial \tau^2}\right|_{\tau=0} \tag{8}$$

where

$$P_\varepsilon(\tau) = \mathrm{Var}(\varepsilon)\,\mathrm{Exp}\,[\tau^2/d_0^2] \tag{9}$$

is the autocorrelation function of ε, and its integral scale or correlation length is $l_s = d_0\sqrt{\pi}/2$. For this autocorrelation, we obtain, as follows from eqns (7)–(9) under the assumption of small fluctuations in ε

$$\langle E \rangle = E(m) - \frac{1}{\pi} l_s^2 \left[\frac{\partial^2 E}{\partial x^2}\right]_{\varepsilon=m}. \tag{10}$$

For a rate formulation, i.e. incremental damage theory, the exact same steps as above will yield

$$\langle \dot{E} \rangle = \dot{E}(\dot{m}) - \frac{1}{\pi} l_s^2 \left[\frac{\partial^2 \dot{E}}{\partial x^2}\right]_{\dot{\varepsilon}=\dot{m}}. \tag{11}$$

Then, from relation 2 in Carmeliet and de Borst (1995), for the purposes of this letter, without claiming that a robust material law formulation results, considering that the rate of damage is expressed as, in general, $\dot{D} = f(E)\langle \dot{E} \rangle$, we obtain

$$\dot{\sigma} = (1 - D_0 - f(E)\varepsilon_0)C\dot{E} - f(E)\frac{1}{\pi}\varepsilon_0 C l_s^2 \frac{\partial^2 \dot{E}}{\partial x^2}. \tag{12}$$

Clearly, from the analogy between eqns (12) and (3) the two length scales in Carmeliet and de Borst (1995) are related. For example, if the *local* formulation of Carmeliet and de Borst (1995) is such that it yields/considers a strain rate field with Gaussian autocorrelation, of correlation length θ, then

$$\theta = l_s = \sqrt{b\pi}l. \tag{13}$$

For the specific formulation where K_0 is a random field, a different relation from eqn (13) holds [for the first equation in (13)] but still θ is related to l. We see no point in investigating the exact relation, since the problem can be formulated more robustly than done by Carmeliet and de Borst (1995). Thus, we see that eqn (3) is redundant and unjustifiable.

Perhaps the "juice" from the equivalence shown above is the following. In the post localization regime, the global stationarity of the strain (rate) will break down. It will most probably become locally stationary, and this has the following important consequence: the length scale in gradient and non-local theories evolves during localization. The exact evolution can be identified by studying the statistical properties of the strain field after localization onset. Such a study can be performed by conditioning the strain field at the localization site, i.e. for a ductile material, or by allowing a crack to form for a quasi-brittle material. These issues are discussed extensively in Frantziskonis (1995), where also further details on the statistical and gradient continuum "equivalence" can be found.

Finally, since this is a scientific communication, the author of this letter feels disturbed by the fact that the paper discussed herein was also published, with minor differences in the correlation functions, in Carmeliet, J. and Hens, H., "Probabilistic nonlocal damage model for continua with random field properties," *J. Engng Mech. Div. ASCE* **120,** 2013–2027

(1994). Further, J. Carmeliet requested and was given a preprint of Frantziskonis (1995), in November 1993, during a workshop; it seems that the results in Frantziskonis (1995) have been misinterpreted, or ignored due to the authors "momentum" as is evident from the present discussion.

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